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BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR IN LOCAL MORREY-TYPE SPACES

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Spaces $L_p(\Omega)$, $0 < p < \infty$

Let $0 < p < \infty$, Ω be a measurable set in R^n, and a function $f : \Omega \to C$. The function f □ $L_p(\Omega)$ if f is measurable on Ω and

$$\|f\|_{L_p(\Omega)} := \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} < \infty$$

Note that if meas$\Omega > 0$, then the conditions f □ $L_p(\Omega)$ and $\|f\|_{L_p(\Omega)} < \infty$ are not equivalent. If, for example, $f:= 1$ on a non-measurable subset G of the set $\Omega \cap B_r$ where r>0is such that meas$(\Omega \cap B_r)>0$, $f:=-1$ on$(\Omega \cap B_r)\backslash G$, and $f := 0$ on $\Omega \setminus B_r$, then f is not measurable on Ω, hence does not belong to $L_p(\Omega)$ for any$0<p<\infty$, but $\|f\|_{L_p(\Omega)}<\infty$.

About Weak Lp-Spaces

One of the real attraction of Weak L_p space is that the subject is sufficiently concrete and yet the spaces have fine structure of importance for applications. Weak L_p spaces are function spaces which are closely related to L_p spaces. The Book by Colin Benett and Robert Sharpley contains a good presentation of Weak L_p but from the point of view of rearrangement function. In the present paper we study the Weak L_p space from the point of view of distribution function.

Next we define this kind of spaces.

For f a measurable function on X, the distribution function of f is the function D_f defined on $[0,\infty)$ as follows:

$$D_f(\lambda) := \mu(\{x \in X: |f(x)| > \lambda\}).$$

The distribution function D_f provides information about the size of f but not about the

behavior of f itself near any given point. For instance, a function on R^n and each of it's translates have the same distribution function. It follows from definition before that D_f is a decreasing function of λ (not necessarily strictly).

Let (X, μ) be a measurable space and f and g be a measurable functions on (X, μ) then D_f enjoy the following properties: For all $\lambda_1, \lambda_2 > 0$:

1. $|g| \leq |f|$ μ-a.e. implies that $D_g \leq D_f$;

2. $D_{cf}(\lambda_1) = D_f\left(\frac{\lambda_1}{|c|}\right)$ for all $c \in \mathbb{C}/\{0\}$
3. $D_{f+g}(\lambda_1 + \lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$
4. $D_{fg}(\lambda_1\lambda_2) \le D_f(\lambda_1) + D_g(\lambda_2)$

Next, let (X, μ) be a measurable space, for $0 < p < \infty$, we consider
$\text{Weak } L_p := \left\{f: \mu(\{x \in X: |f(x)| > \lambda\}) \le \left(\frac{C}{\lambda}\right)^p\right\}$, for some $C > 0$.
Let $f \in L_1^{loc}(\mathbb{R}^n)$. The fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \qquad 0 \le \alpha \le n$$

where $|B(x, t)|$ is the Lebesgue measure of the ball $B(x,t)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The operator M_α plays an important role in real and harmonic analysis.

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $M_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 and defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le n$, a function f □M if f □ $L_p^{loc}(\mathbb{R}^n)$ and

$$||f||_{\mathcal{M}_{p,\lambda}} \equiv ||f||_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p} ||f||_{L_p(B(x,r))} < \infty$$

Also by $WM_{p,\lambda}$ we denote the weak Morrey space of all functions f □ $WL_p^{loc}(\mathbb{R}^n)$ for which

$$||f||_{WM_{p,\lambda}} \equiv ||f||_{WM_{p,\lambda}(R^n)} = \text{SUP}_{x\square R^n,\, r>0}\ r^{-\lambda/p} ||f||_{WLp(B(x,r))} < \infty,$$

Where $WL_p(B(x,r))$ denotes the Weak Lp-space of measurable functions f for which

$$||f||_{WL_p(B(x,r))} \equiv \left|\left|f\chi_{B(x,r)}\right|\right|_{WL_p(\mathbb{R}^n)}$$
$$= \sup_{t>0} t|\{y \in B(x,r): |f(y)| > t\}|^{1/p}$$
$$= \sup_{t>0} t^{1/p} \left(f\chi_{B(x,r)}\right)^*(y) < \infty.$$

Here $g^\square$ denotes the non-increasing rearrangement of the function g.

Spanne and Adams studied boundedness of the fractional maximal operator M_α for $0 < \alpha < n$ in Morrey spaces $M_{p,\lambda}$. Later on Chiarenza and Frasca studied boundedness of the maximal operator M in these spaces. Their results, together with the classical results for L_p-spaces, can be summarized as follows.

Theorem 1.

(1) Let $1<p_1 <p_2 \le\infty$ and $0<\alpha<n$. Then M_α is bounded from $M_{p_1,\lambda}$ to $M_{p_2,\lambda}$ if and only if

$$\alpha \le n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ and } \lambda = \left(n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \alpha\right)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)^{-1}.$$

(2) Let $1<p_2 \le\infty$ and $0<\alpha<n$. Then M_α is bounded from $M_{1,\lambda}$ to $WM_{p_2,\lambda}$

if and only if $\alpha \le n\left(1-\frac{1}{p_2}\right)$ and $\lambda = \left(n\left(1-\frac{1}{p_2}\right)-\alpha\right)\left(1-\frac{1}{p_2}\right)^{-1}$.

(3) Let $1<p\le\infty$. Then M is bounded from $M_{p,\lambda}$ to $M_{p,\lambda}$ for all $0 \le \lambda \le n$.

(4) The operator M is bounded from $M_{1,\lambda}$ to $WM_{1,\lambda}$ for all $0 \le \lambda \le n$.

If in the place of the power function $r^{-\lambda/p}$ in the definition of $M_{p,\lambda}$ we consider any positive measurable weight function w defined on $(0, \infty)$, then it becomes the Morrey-type space $M_{p,w}$. T. Mizuhara, E. Nakai and V.S. Guliyev generalized Theorem 1.1 and obtained sufficient conditions on weights w_1 and w_2 ensuring boundedness of the maximal operator M and the fractional maximal operator M_α for the limiting case from

M_{p_1,w_1} to M_{p_2,w_2}.